11.2 SERIES

DEF'N:

GIVEN INFINITE SERIES $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$

LET $S_n = \sum_{i=1}^{n} a_i$

IF $\{S_n\}$ CONVERGES AND $\lim_{n \to \infty} S_n = S$

THEN WE SAY $\sum_{i=1}^{\infty} a_i$ CONVERGES AND $\sum_{i=1}^{\infty} a_i = S$

IF $\{S_n\}$ DIVERGES, THEN WE SAY

$\sum_{i=1}^{\infty} a_i$ DIVERGES

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eg. $\sum_{i=1}^{\infty} \frac{1}{i^2}$

1ST PARTIAL SUM $= S_1 = \sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1$

2ND PARTIAL SUM $= S_2 = \sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}$

3RD PARTIAL SUM $= S_3 = \sum_{i=1}^{3} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36}$

$\left\{1, \frac{5}{4}, \frac{49}{36}, \ldots\right\}$

SEQUENCE OF PARTIAL SUMS

DOES $\sum_{i=1}^{\infty} 1$ CONV/DIV? $1+1+1+1+\cdots$

$$S_1 = \sum_{i=1}^{1} 1 = 1$$

$$S_2 = \sum_{i=1}^{2} 1 = 1+1 = 2$$

$$S_3 = \sum_{i=1}^{3} 1 = 1+1+1 = 3$$

$$S_n = \sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ TERMS}} = 1 \cdot n = n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

$\{S_n\}$ DIV, SO $\sum_{i=1}^{\infty} 1$ DIV

NOTE: $\{a_n\} = \{1\}$ CONV

BUT THIS IS UNRELATED TO WHETHER $\sum_{i=1}^{\infty} a_i$ CONV/DIV

DOES $\sum_{i=1}^{\infty} (-1)^i$ CONV / DIV ?

$$-1+1-1+1-1+1 \ldots$$

$$\underbrace{-1+1}_{=0}\;\underbrace{-1+1}_{=0}\;\underbrace{-1+1}_{=0} \longrightarrow = 0$$

$$0+0+0+\ldots \longrightarrow 0$$

$$-1+\underbrace{1-1}_{=0}+\underbrace{1-1}_{=0}+\underbrace{1-1}_{=0} \ldots$$

$$-1+0+0+0 \longrightarrow -1$$

$S_1 = -1$

$S_2 = -1+1 = 0$

$S_3 = -1+1-1 = -1$

$S_4 = -1+1-1+1 = 0$

$\{S_n\} = \{-1, 0, -1, 0, -1, 0, \ldots\}$

$\lim_{n\to\infty} S_n$ DNE $\longrightarrow$ $\{S_n\}$ DIV

SO ~~$\sum_{i=1}^{\infty} a_i$ DIVERGES~~

$\sum_{i=1}^{\infty} (-1)^i$ DIVERGES

DOES $\sum_{i=1}^{\infty} \frac{2}{i(i+2)}$ CONV/DIV? $\frac{2}{1\cdot 3} + \frac{2}{2\cdot 4} + \frac{2}{3\cdot 5} + \frac{2}{4\cdot 6} + \cdots$

$$= \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2} \right)$$ BY PARTIAL FRACTIONS DECOMPOSITION

$$S_1 = \left(\frac{1}{1} - \frac{1}{3}\right)$$

$$S_2 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$S_3 = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right)$$

TELESCOPING SERIES - LATER TERMS CANCEL PRIOR TERMS

$$S_4 = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$$

$$S_5 = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7}\right)$$

$$S_6 = \frac{1}{1} + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}$$

$$S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \qquad n \geq 2$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = 1 + \frac{1}{2} - 0 - 0 = \frac{3}{2}$$

SO $\sum_{i=1}^{\infty} \frac{2}{i(i+2)}$ CONV TO $\frac{3}{2}$

GEOMETRIC SERIES

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + ar^3 + ar^4 + \dots$$

$*r \quad *r \quad *r \quad *r$ ← MULTIPLY BY SAME CONSTANT TO GO FROM EACH TERM TO THE NEXT → GEOMETRIC

ASSUME $a \neq 0$

IF $r = 1$, $\sum_{i=1}^{\infty} a \cdot 1^{i-1} = \sum_{i=1}^{\infty} a$

$a + a + a + \dots$

$S_n = \sum_{i=1}^{n} a = na$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty$ OR $-\infty$

IE. IF $r=1$, $\sum_{i=1}^{\infty} ar^{i-1}$ DIV

IF $r \neq 1$, $\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \dots + ar^{n-1} = S_n$

$ar + ar^2 + ar^3 + \dots + ar^n = rS_n$

$(1-r)S_n = S_n - rS_n = a - ar^n$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}(1-0)$$

$= \frac{a}{1-r}$ IF $|r| < 1$